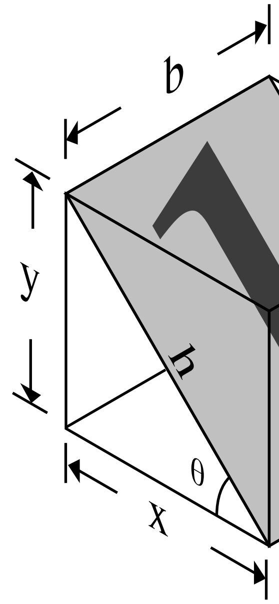
We wish to determine the stresses acting on a plane with an arbitrary orientation. Let the angle between this plane and the plane normal to the maximum stress (σ_1) be given as θ.



Next, we note that as we are working with a two dimensional stress system, we can adjust the size of the third dimension (b) such that the area of the plane in question is equal to one.

This means that b * h = 1 or that b = 1 / h .

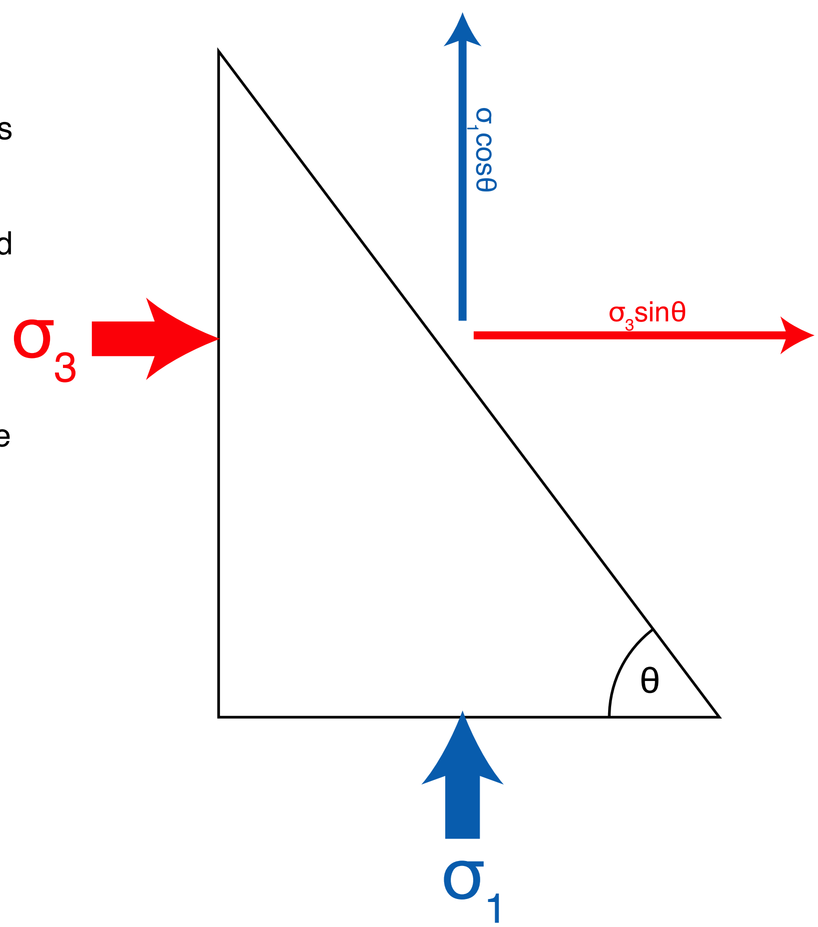
Because cos θ = x / h, x = h cos θ, and so the bottom of the box, the area over which σ_1 is acting, is equal to x * b, or cos θ.

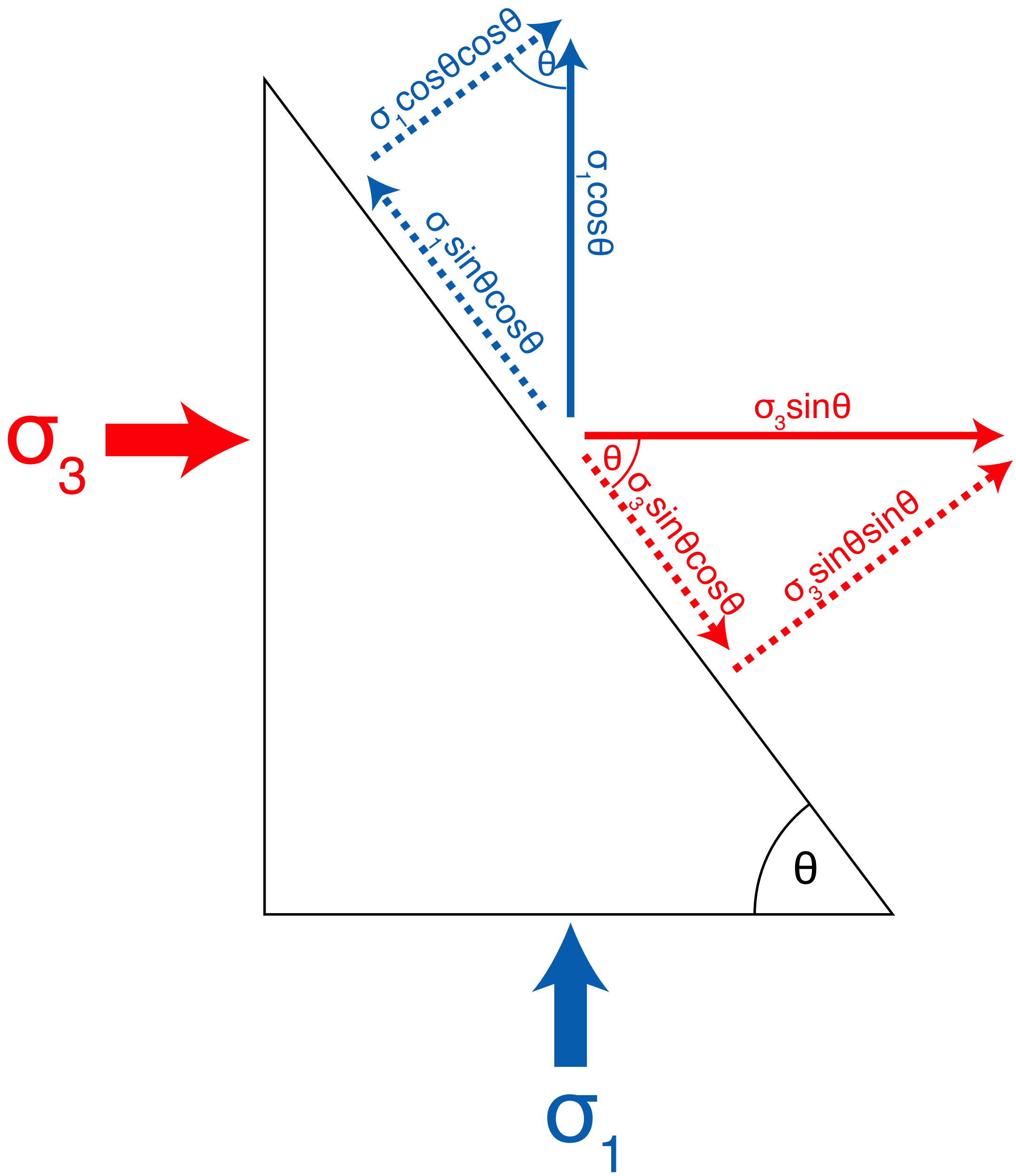
Because sin θ = y / h, y = h sin θ, and so the side of the box, the area over which σ_3 is acting, is equal to y * b, or sin θ.

Knowing these areas, we can get the forces acting on our plane by multiplying each stress times the area over which it is acting.

We could, if we wished, now add these forces together to get the net force acting on our plane. Note that we could not add, or resolve, the stresses. We needed to convert them to forces, by multiplying times the areas over which they acted.

Forces are vectors, and so can be resolved into components in various directions. For our purposes, it is useful to resolve them into those components which are normal (perpendicular) to the plane in question, called “normal forces,” and those which are parallel to the plane in question, which are called “shear forces.”



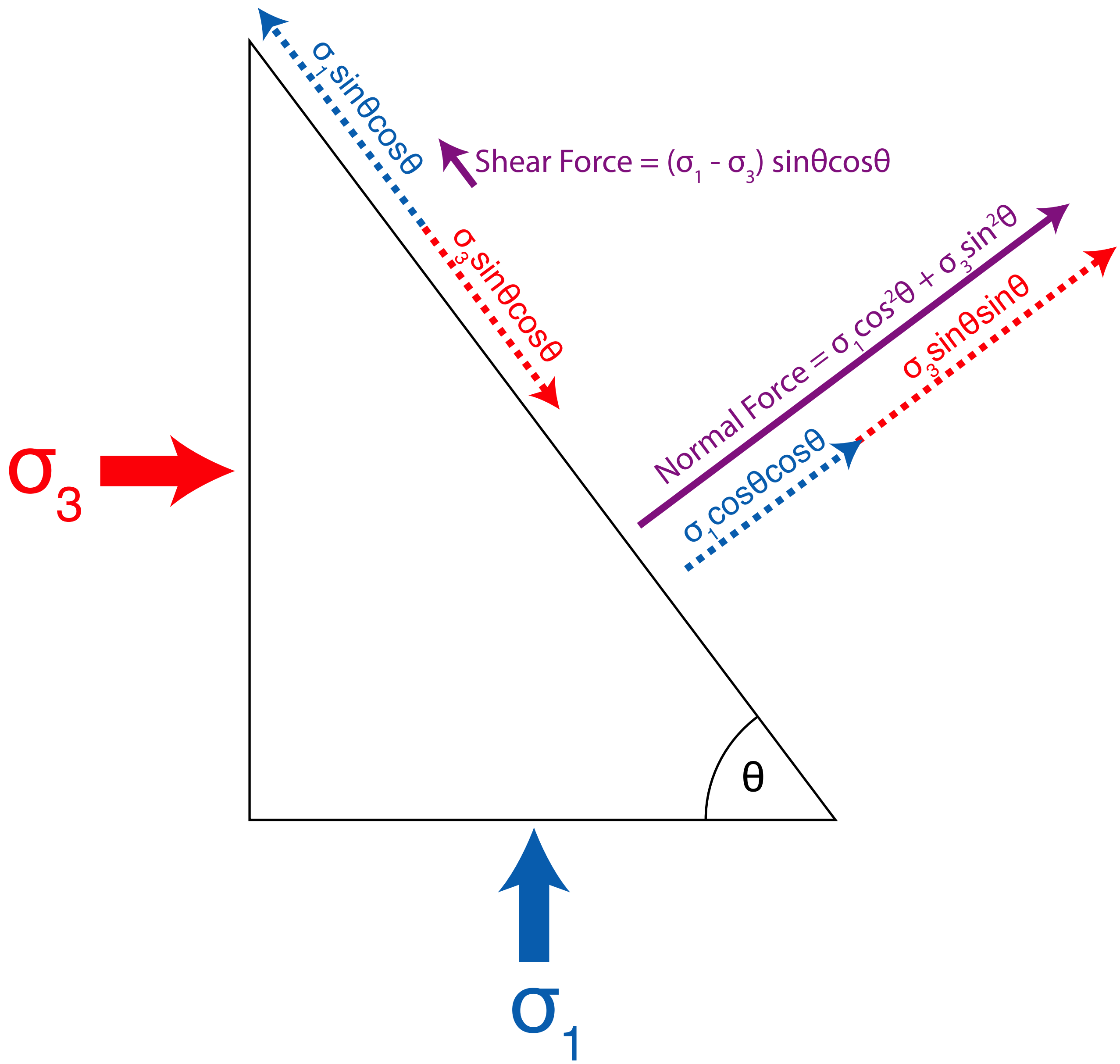


First, we need to establish our geometrical relationships:

The red θ is equal to the black θ because of “parallel lines cut by a straight line” and the blue θ must equal the black θ because the other angles in the blue triangle are 90° and (180° - 90° -θ).

Then we can resolve each of the solid colored vectors into two components, one normal to, and the other parallel to, the plane in question.

These components can then be added, recalling that to add vectors you put the tail of one on the head of the other.



Now that we resolved our forces into components, added the components and determined the resultant forces acting normal and parallel to our plane, we can convert these forces back into stresses by dividing by the area over which they act. But, aha! We started by stipulating that this area would be equal to one... (weren't we clever?).

Shear stress $\sigma_\tau = (\sigma_1 - \sigma_3) \sin\theta \cos\theta$ Or, using earlier results: $\sigma_\tau = [(\sigma_1 - \sigma_3) / 2] \sin 2\theta$

Normal stress $\sigma_n = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$

But, from earlier, we know that $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

Rearranging: $\cos^2\theta = (1 + \cos 2\theta) / 2$ and $\sin^2\theta = (1 - \cos 2\theta) / 2$

Giving us $\sigma_n = [\sigma_1 (1 + \cos 2\theta) / 2] + [\sigma_3 (1 - \cos 2\theta)] = (\sigma_1 + \sigma_3) / 2 + [(\sigma_1 - \sigma_3) / 2] \cos 2\theta$

As can be shown, these are the parametric equations for a circle, in terms of 2θ, with a radius of $(\sigma_1 - \sigma_3) / 2$ and a center at $(\sigma_1 + \sigma_3) / 2$.

Recall that the angle θ is the angle between the plane in question and the plane which is normal to the σ_1 direction. It is zero when the plane is being acted on by only σ_1, and will be 90° (that is, 2θ will be 180°) when the plane is being acted on by only σ_3.

This is illustrated below: